Dynamical systems method for solving linear finite-rank operator equations

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Abstract

A version of the Dynamical Systems Method (DSM) for solving ill-conditioned linear algebraic systems is studied in this paper. An *a priori* and *a posteriori* stopping rules are justified. An iterative scheme is constructed for solving ill-conditioned linear algebraic systems.

Keywords. Ill-posed problems, Dynamical Systems Method, Variational Regularization

1 Introduction

We want to solve stably the equation

$$Au = f, \tag{1}$$

where A is a linear bounded operator in a real Hilbert space H. We assume that (1) has a solution, possibly nonunique, and denote by y the unique minimal-norm solution to (1), $y \perp \mathcal{N} := \mathcal{N}(A) := \{u : Au = 0\}$, $Ay = f$. We assume that the range of A, $R(A)$, is not closed, so problem (1) is ill-posed. Let f_δ, $\|f - f_\delta\| \le \delta$, be the noisy data. We want to construct a stable approximation of y, given $\{\delta, f_\delta, A\}$. There are many methods for doing this, see, e.g., [4]–[6], [7], [14], [15], to mention some (of the many) books, where variational regularization, quasisolutions, quasiinversion, and iterative regularization are studied, and [7]-[12], where the Dynamical Systems Method (DSM) is studied systematically (see also [1], [14], [13], and references therein for related results). The basic new results of this paper are: 1) a new version of the DSM for solving equation (1) is justified; 2) a stable method for solving equation (1) with noisy data by the DSM is given; a priori and a posteriori stopping rules are proposed and justified; 3) an iterative method for solving linear ill-conditioned algebraic systems, based on the proposed version of DSM, is formulated; its convergence is proved; 4) numerical results are given; these results show that the proposed

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method yields a good alternative to some of the standard methods (e.g., to variational regularization, Landweber iterations, and some other methods).

The DSM version we study in this paper consists of solving the Cauchy problem

$$\dot{u}(t) = -P(Au(t) - f), \quad u(0) = u_0, \quad u_0 \perp \mathcal{N}, \quad \dot{u} := \frac{du}{dt}, \tag{2}$$

and proving the existence of the limit $\lim_{t\to\infty} u(t) = u(\infty)$, and the relation $u(\infty) = y$, i.e.,

$$\lim_{t\to\infty} \|u(t) - y\| = 0. \tag{3}$$

Here P is a bounded operator such that $T := PA \geq 0$ is selfadjoint, $\mathcal{N}(T) = \mathcal{N}(A)$.

For any linear (not necessarily bounded) operator A there exists a bounded operator P such that $T = PA \geq 0$. For example, if $A = U|A|$ is the polar decomposition of A, then $|A| := (A^*A)^{\frac{1}{2}}$ is a selfadjoint operator, $T := |A| \geq 0$, U is a partial isometry, $\|U\| = 1$, and if $P := U^*$, then $\|P\| = 1$ and $PA = T$. Another choice of P, namely, $P = (A^*A + aI)^{-1}A^*$, $a = const > 0$, is used in Section 3.

If the noisy data f_δ are given, $\|f_\delta - f\| \leq \delta$, then we solve the problem

$$\dot{u}_\delta(t) = -P(Au_\delta(t) - f_\delta), \quad u_\delta(0) = u_0, \tag{4}$$

and prove that, for a suitable stopping time t_δ, and $u_\delta := u_\delta(t_\delta)$, one has

$$\lim_{\delta\to 0} \|u_\delta - y\| = 0. \tag{5}$$

An *a priori* and an *a posteriori* methods for choosing t_δ are given.

In Section 2 these results are formulated and recipes for choosing t_δ are proposed. In Section 3 a numerical example is presented.

2 Formulation and results

Suppose $A : H \to H$ is a linear bounded operator in a real Hilbert space H. Assume that equation (1) has a solution not necessarily unique. Denote by y the unique minimal-norm solution i.e., $y \perp \mathcal{N} := \mathcal{N}(A)$. Consider the DSM (2) where $u_0 \perp \mathcal{N}$ is arbitrary. Denote

$$T := PA, \quad Q := AP. \tag{6}$$

The unique solution to (2) is

$$u(t) = e^{-tT}u_0 + e^{-tT}\int_0^t e^{sT}dsPf. \tag{7}$$

Let us first show that any ill-posed linear equation (1) with exact data can be solved by the DSM.

2.1 Exact data

The following result is known (see [7]) but a short proof is included for completeness.

Theorem 1 *Suppose $u_0 \perp \mathcal{N}$ and $T^* = T \geq 0$. Then problem* (2) *has a unique solution defined on* $[0, \infty)$, *and* $u(\infty) = y$, *where* $u(\infty) = \lim_{t\to\infty} u(t)$.

Proof. Denote $w := u(t) - y$, $w_0 := w(0) = u_0 - y$. Note that $w_0 \perp \mathcal{N}$. One has

$$\dot{w} = -Tw, \quad T := PA, \quad w(0) = u_0 - y. \tag{8}$$

The unique solution to (8) is $w = e^{-tT} w_0$. Thus,

$$\|w\|^2 = \int_0^{\|T\|} e^{-2t\lambda} d\langle E_\lambda w_0, w_0\rangle.$$

where $\langle u, v\rangle$ is the inner product in H, and E_λ is the resolution of the identity of T. Thus,

$$\|w(\infty)\|^2 = \lim_{t\to\infty} \int_0^{\|T\|} e^{-2t\lambda} d\langle E_\lambda w_0, w_0\rangle = \|P_{\mathcal{N}} w_0\|^2 = 0,$$

where $P_{\mathcal{N}} = E_0 - E_{-0}$ is the orthogonal projector onto $\mathcal{N}$. Theorem 1 is proved. □

2.2 Noisy data f_δ

Let us solve stably equation (1) assuming that f is not known, but f_δ, the noisy data, are known, where $\|f_\delta - f\| \leq \delta$. Consider the following DSM

$$\dot{u}_\delta = -P(Au_\delta - f_\delta), \quad u_\delta(0) = u_0. \tag{9}$$

Denote

$$w_\delta := u_\delta - y, \quad T := PA, \quad w_\delta(0) = w_0 := u_0 - y \in \mathcal{N}^\perp.$$

Let us prove the following result:

Theorem 2 *If $T = T^* \geq 0$, $\lim_{\delta\to 0} t_\delta = \infty$, $\lim_{\delta\to 0} t_\delta\delta = 0$, and $w_0 \in \mathcal{N}^\perp$, then*

$$\lim_{\delta\to 0} \|w_\delta(t_\delta)\| = 0.$$

Proof. One has

$$\dot{w}_\delta = -Tw_\delta + \zeta_\delta, \quad \zeta_\delta = P(f_\delta - f), \quad \|\zeta_\delta\| \leq \|P\|\delta. \tag{10}$$

The unique solution of equation (10) is

$$w_\delta(t) = e^{-tT} w_\delta(0) + \int_0^t e^{-(t-s)T} \zeta_\delta ds.$$

Let us show that $\lim_{\delta\to 0}\|w_\delta(t_\delta)\| = 0$. One has

$$\lim_{t\to\infty}\|w_\delta(t)\| \le \lim_{t\to\infty}\|e^{-tT}w_\delta(0)\| + \lim_{t\to\infty}\left\|\int_0^t e^{-(t-s)T}\zeta_\delta ds\right\|. \tag{11}$$

Let E_λ be the resolution of identity corresponding to T. One uses the spectral theorem and gets:

$$\begin{aligned}\int_0^t e^{-(t-s)T}ds\zeta_\delta &= \int_0^t\int_0^{\|T\|} dE_\lambda\zeta_\delta e^{-(t-s)\lambda}ds\\ &= \int_0^{\|T\|} e^{-t\lambda}\frac{e^{t\lambda}-1}{\lambda}dE_\lambda\zeta_\delta = \int_0^{\|T\|}\frac{1-e^{-t\lambda}}{\lambda}dE_\lambda\zeta_\delta.\end{aligned} \tag{12}$$

Note that

$$0 \le \frac{1-e^{-t\lambda}}{\lambda} \le t, \quad \forall\lambda > 0, \quad t \ge 0, \tag{13}$$

since $1 - x \le e^{-x}$ for $x \ge 0$. From (12) and (13), one obtains

$$\begin{aligned}\left\|\int_0^t e^{-(t-s)T}ds\zeta_\delta\right\|^2 &= \int_0^{\|T\|}\left|\frac{1-e^{-t\lambda}}{\lambda}\right|^2 d\langle E_\lambda\zeta_\delta,\zeta_\delta\rangle\\ &\le t^2\int_0^{\|T\|} d\langle E_\lambda\zeta_\delta,\zeta_\delta\rangle\\ &= t^2\|\zeta_\delta\|^2.\end{aligned} \tag{14}$$

Since $\|\zeta_\delta\| \le \|P\|\delta$, from (11) and (14), one gets

$$\lim_{\delta\to 0}\|w_\delta(t_\delta)\| \le \lim_{\delta\to 0}\Big(\|e^{-t_\delta T}w_\delta(0)\| + t_\delta\delta\|P\|\Big) = 0.$$

Here we have used the relation:

$$\lim_{\delta\to 0}\|e^{-t_\delta T}w_\delta(0)\| = \|P_{\mathcal{N}}w_0\| = 0,$$

and the last equality holds because $w_0 \in \mathcal{N}^\perp$. Theorem 2 is proved. $\Box$

From Theorem 2, it follows that the relation

$$t_\delta = \frac{C}{\delta^\gamma}, \quad \gamma = \text{const}, \quad \gamma \in (0,1)$$

where $C > 0$ is a constant, can be used as an *a priori* stopping rule, i.e., for such t_δ one has

$$\lim_{\delta\to 0}\|u_\delta(t_\delta) - y\| = 0. \tag{15}$$

2.3 Discrepancy principle

In this section we assume that A is a linear finite-rank operator. Thus, it is a linear bounded operator. Let us consider equation (1) with noisy data f_δ, and a DSM of the form

$$\dot{u}_\delta = -PAu_\delta + Pf_\delta, \quad u_\delta(0) = u_0. \tag{16}$$

for solving this equation. Equation (16) has been used in Section 2.2. Recall that y denotes the minimal-norm solution of equation (1). Example of a choice of P is given in Section 3.

Theorem 3 *Let $T := PA$, $Q := AP$. Assume that $\|Au_0 - f_\delta\| > C\delta$, $Q = Q^* \geq 0$, $T^* = T \geq 0$, T is a finite-rank operator. Let $\mathcal{N}(T) =: \mathcal{N}$. Note that $\mathcal{N}(T) = \mathcal{N}(A)$. The solution t_δ to the equation*

$$h(t) := \|Au_\delta(t) - f_\delta\| = C\delta, \quad C = const, \quad C \in (1,2), \tag{17}$$

does exist, is unique, and

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0, \tag{18}$$

where y is the unique minimal-norm solution to (1).

Proof. Denote

$$v_\delta(t) := Au_\delta(t) - f_\delta, \quad w(t) := u(t) - y, \quad w_0 := u_0 - y.$$

One has

$$\begin{aligned}
\frac{d}{dt}\|v_\delta(t)\|^2 &= 2\langle A\dot{u}_\delta(t), Au_\delta(t) - f_\delta\rangle \\
&= 2\langle A[-P(Au_\delta(t) - f_\delta)], Au_\delta(t) - f_\delta\rangle \\
&= -2\langle AP(Au_\delta - f_\delta), Au_\delta - f_\delta\rangle \leq 0.
\end{aligned} \tag{19}$$

where the last inequality holds because $AP = Q \geq 0$. Thus, $\|v_\delta(t)\|$ is a nonincreasing function.

Let us prove that equation (17) has a solution for $C \in (1,2)$. One has the following commutation formulas:

$$e^{-sT}P = Pe^{-sQ}, \quad Ae^{-sT} = e^{-sQ}A.$$

Using these formulas and the representation

$$u_\delta(t) = e^{-tT}u_0 + \int_0^t e^{-(t-s)T}Pf_\delta ds,$$

one gets:

$$\begin{aligned}
v_\delta(t) &= Au_\delta(t) - f_\delta \\
&= Ae^{-tT}u_0 + A\int_0^t e^{-(t-s)T}Pf_\delta ds - f_\delta \\
&= e^{-tQ}Au_0 + e^{-tQ}\int_0^t e^{sQ}dsQf_\delta - f_\delta \\
&= e^{-tQ}A(u_0 - y) + e^{-tQ}f + e^{-tQ}(e^{tQ} - I)f_\delta - f_\delta \\
&= e^{-tQ}Aw_0 - e^{-tQ}f_\delta + e^{-tQ}f = e^{-tQ}Au_0 - e^{-tQ}f_\delta.
\end{aligned} \tag{20}$$

Note that

$$\lim_{t\to\infty} e^{-tQ}Aw_0 = \lim_{t\to\infty} Ae^{-tT}w_0 = AP_{\mathcal{N}}w_0 = 0.$$

Here the continuity of A and the following relation

$$\lim_{t\to\infty} e^{-tT}w_0 = \lim_{t\to\infty} \int_0^{\|T\|} e^{-st} dE_s w_0 = (E_0 - E_{-0})w_0 = P_{\mathcal{N}}w_0,$$

were used. Therefore,

$$\lim_{t\to\infty} \|v_\delta(t)\| = \lim_{t\to\infty} \|e^{-tQ}(f - f_\delta)\| \le \|f - f_\delta\| \le \delta, \tag{21}$$

where $\|e^{-tQ}\| \le 1$ because $Q \ge 0$. The function $h(t)$ is continuous on $[0,\infty)$, $h(0) = \|Au_0 - f_\delta\| > C\delta$, $h(\infty) \le \delta$. Thus, equation (17) must have a solution t_δ.

Let us prove the uniqueness of t_δ. If t_δ is non-unique, then without loss of generality we can assume that there exists $t_1 > t_\delta$ such that $\|Au_\delta(t_1) - f_\delta\| = C\delta$. Since $\|v_\delta(t)\|$ is nonincreasing and $\|v_\delta(t_\delta)\| = \|v_\delta(t_1)\|$, one has

$$\|v_\delta(t)\| = \|v_\delta(t_\delta)\|, \quad \forall t \in [t_\delta, t_1].$$

Thus,

$$\frac{d}{dt}\|v_\delta(t)\|^2 = 0, \quad \forall t \in (t_\delta, t_1). \tag{22}$$

Using (19) and (22) one obtains

$$\|\sqrt{AP}(Au_\delta(t) - f_\delta)\|^2 = \langle AP(Au_\delta(t) - f_\delta), Au_\delta(t) - f_\delta\rangle = 0, \quad \forall t \in [t_\delta, t_1],$$

where $\sqrt{AP} = Q^{\frac{1}{2}} \ge 0$ is well defined since $Q = Q^* \ge 0$. This implies $Q^{\frac{1}{2}}(Au_\delta - f_\delta) = 0$. Thus

$$Q(Au_\delta(t) - f_\delta) = 0, \quad \forall t \in [t_\delta, t_1]. \tag{23}$$

From (20) one gets:

$$v_\delta(t) = Au_\delta(t) - f_\delta = e^{-tQ}Au_0 - e^{-tQ}f_\delta. \tag{24}$$

Since $Qe^{-tQ} = e^{-tQ}Q$ and e^{-tQ} is an isomorphism, equalities (23) and (24) imply

$$Q(Au_0 - f_\delta) = 0.$$

This and (24) imply

$$AP(Au_\delta(t) - f_\delta) = e^{-tQ}(QAu_0 - Qf_\delta) = 0, \quad t \ge 0.$$

This and (19) imply

$$\frac{d}{dt}\|v_\delta\|^2 = 0, \quad t \ge 0. \tag{25}$$

Consequently,

$$C\delta < \|Au_\delta(0) - f_\delta\| = \|v_\delta(0)\| = \|v_\delta(t_\delta)\| = \|Au_\delta(t_\delta) - f_\delta\| = C\delta.$$

This is a contradiction which proves the uniqueness of t_δ.

Let us prove (18). First, we have the following estimate:

$$\begin{aligned}\|Au(t_\delta) - f\| &\le \|Au(t_\delta) - Au_\delta(t_\delta)\| + \|Au_\delta(t_\delta) - f_\delta\| + \|f_\delta - f\| \\ &\le \left\|e^{-t_\delta Q}\int_0^{t_\delta} e^{sQ}Q ds\right\| \|f_\delta - f\| + C\delta + \delta,\end{aligned} \tag{26}$$

where $u(t)$ solves (2) and $u_\delta(t)$ solves (9). One uses the inequality:

$$\left\|e^{-t_\delta Q}\int_0^{t_\delta} e^{sQ}Q ds\right\| = \|I - e^{-t_\delta Q}\| \le 2,$$

and concludes from (26), that

$$\lim_{\delta\to 0} \|Au(t_\delta) - f\| = 0. \tag{27}$$

Secondly, we claim that

$$\lim_{\delta\to 0} t_\delta = \infty.$$

Assume the contrary. Then there exist $t_0 > 0$ and a sequence $(t_{\delta_n})_{n=1}^\infty$, $t_{\delta_n} < t_0$, such that

$$\lim_{n\to\infty} \|Au(t_{\delta_n}) - f\| = 0. \tag{28}$$

Analogously to (19), one proves that

$$\frac{d}{dt}\|v\|^2 \le 0,$$

where $v(t) := Au(t) - f$. Thus, $\|v(t)\|$ is nonincreasing. This and (28) imply the relation $\|v(t_0)\| = \|Au(t_0) - f\| = 0$. Thus,

$$0 = v(t_0) = e^{-t_0 Q}A(u_0 - y).$$

This implies $A(u_0 - y) = e^{t_0 Q}e^{-t_0 Q}A(u_0 - y) = 0$, so $u_0 - y \in \mathcal{N}$. Since $u_0 - y \in \mathcal{N}^\perp$, it follows that $u_0 = y$. This is a contradiction because

$$C\delta \le \|Au_0 - f_\delta\| = \|f - f_\delta\| \le \delta, \quad 1 < C < 2.$$

Thus,

$$\lim_{\delta\to 0} t_\delta = \infty. \tag{29}$$

Let us continue the proof of (18). From (20) and the relation $\|Au_\delta(t_\delta) - f_\delta\| = C\delta$, one has

$$\begin{aligned}C\delta t_\delta &= \|t_\delta e^{-t_\delta Q}Aw_0 - t_\delta e^{-t_\delta Q}(f_\delta - f)\| \\ &\le \|t_\delta e^{-t_\delta Q}Aw_0\| + \|t_\delta e^{-t_\delta Q}(f_\delta - f)\| \\ &\le \|t_\delta e^{-t_\delta Q}Aw_0\| + t_\delta\delta.\end{aligned} \tag{30}$$

We claim that

$$\lim_{\delta\to 0} t_\delta e^{-t_\delta Q} A w_0 = \lim_{\delta\to 0} t_\delta A e^{-t_\delta T} w_0 = 0. \tag{31}$$

Note that (31) holds if $T \geq 0$ has finite rank, and $w_0 \in \mathcal{N}^\perp$. It also holds if $T \geq 0$ is compact and the Fourier coefficients $w_{0j} := \langle w_0, \phi_j \rangle$, $T\phi_j = \lambda_j \phi_j$, decay sufficiently fast. In this case

$$\|Ae^{-tT} w_0\|^2 \leq \|T^{\frac{1}{2}} e^{-tT} w_0\|^2 = \sum_{j=1}^{\infty} \lambda_j e^{-2\lambda_j t} |w_{0j}|^2 := S = o(\frac{1}{t^2}), \quad t \to \infty,$$

provided that $\sum_{j=1}^{\infty} |w_{0j}| \lambda_j^{-2} < \infty$. Indeed $S = \sum_{\lambda_j \leq \frac{1}{t^{\frac{2}{3}}}} + \sum_{\lambda_j > \frac{1}{t^{\frac{2}{3}}}} := S_1 + S_2$. One has

$$S_1 \leq \frac{1}{t^2} \sum_{\lambda_j \leq t^{-\frac{2}{3}}} \frac{|w_{0j}|^2}{\lambda_j^2} = o(\frac{1}{t^2}), \quad S_2 \leq c e^{-2t^{\frac{1}{3}}} = o(\frac{1}{t^2}), \quad t \to \infty,$$

where $c > 0$ is a constant.

From (31) and (30), one gets

$$0 \leq \lim_{\delta\to 0} (C-1)\delta t_\delta \leq \lim_{\delta\to 0} \|t_\delta e^{-t_\delta Q} A w_0\| = 0.$$

Thus,

$$\lim_{\delta\to 0} \delta t_\delta = 0 \tag{32}$$

Now (18) follows from (29), (32) and Theorem 2. Theorem 3 is proved. □

2.4 An iterative scheme

Let us solve stably equation (1) assuming that f is not known, but f_δ, the noisy data, are known, where $\|f_\delta - f\| \leq \delta$. Consider the following dicrete version of the DSM:

$$u_{n+1,\delta} = u_{n,\delta} - hP(Au_{n,\delta} - f_\delta), \quad u_{\delta,0} = u_0. \tag{33}$$

Let us denote $u_n := u_{n,\delta}$ when $\delta \neq 0$, and set

$$w_n := u_n - y, \quad T := PA, \quad w_0 := u_0 - y \in \mathcal{N}^\perp.$$

Let $n = n_\delta$ be the stopping rule for iterations (33). Let us prove the following result:

Theorem 4 *Assume that $T = T^* \geq 0$, $h\|T\| < 2$, $\lim_{\delta\to 0} n_\delta h = \infty$, $\lim_{\delta\to 0} n_\delta h\delta = 0$, and $w_0 \in \mathcal{N}^\perp$. Then*

$$\lim_{\delta\to 0} \|w_{n_\delta}\| = \lim_{\delta\to 0} \|u_{n_\delta} - y\| = 0. \tag{34}$$

Proof. One has

$$w_{n+1} = w_n - hTw_n + h\zeta_\delta, \quad \zeta_\delta = P(f_\delta - f), \quad \|\zeta_\delta\| \le \|P\|\delta, \quad w_0 = u_0 - y. \tag{35}$$

The unique solution of equation (35) is

$$w_{n+1} = (I - hT)^{n+1} w_0 + h\sum_{i=0}^{n} (I - hT)^i \zeta_\delta.$$

Let us show that $\lim_{\delta\to 0} \|w_{n_\delta}\| = 0$. One has

$$\|w_n\| \le \|(I - hT)^n w_0\| + \left\| h\sum_{i=0}^{n-1} (I - hT)^i \zeta_\delta \right\|. \tag{36}$$

Let E_λ be the resolution of identity corresponding to T. One uses the spectral theorem and gets:

$$\begin{aligned} h\sum_{i=0}^{n-1} (I - hT)^i &= h\sum_{i=0}^{n-1} \int_0^{\|T\|} (1 - h\lambda)^i dE_\lambda \\ &= h\int_0^{\|T\|} \frac{1 - (1 - \lambda h)^n}{1 - (1 - h\lambda)} dE_\lambda = \int_0^{\|T\|} \frac{1 - (1 - \lambda h)^n}{\lambda} dE_\lambda. \end{aligned} \tag{37}$$

Note that

$$0 \le \frac{1 - (1 - h\lambda)^n}{\lambda} \le hn, \quad \forall \lambda > 0, \quad t \ge 0, \tag{38}$$

since $1 - (1 - \alpha)^n \le \alpha n$ for all $\alpha \in [0, 2]$. From (37) and (38), one obtains

$$\begin{aligned} \left\| h\sum_{i=0}^{n-1} (I - hT)^i \zeta_\delta \right\|^2 &= \int_0^{\|T\|} \left| \frac{1 - (1 - \lambda h)^n}{\lambda} \right|^2 d\langle E_\lambda \zeta_\delta, \zeta_\delta \rangle \\ &\le (hn)^2 \int_0^{\|T\|} d\langle E_\lambda \zeta_\delta, \zeta_\delta \rangle \\ &= (nh)^2 \|\zeta_\delta\|^2. \end{aligned} \tag{39}$$

Since $\|\zeta_\delta\| \le \|P\|\delta$, from (36) and (39), one gets

$$\lim_{\delta\to 0} \|w_{n_\delta}\| \le \lim_{\delta\to 0} \left(\|(I - hT)^{n_\delta} w_\delta(0)\| + hn_\delta \delta \|P\| \right) = 0.$$

Here we have used the relation:

$$\lim_{\delta\to 0} \|(I - hT)^{n_\delta} w_\delta(0)\| = \|P_{\mathcal{N}} w_0\| = 0,$$

and the last equality holds because $w_0 \in \mathcal{N}^\perp$. Theorem 4 is proved. □

From Theorem 4, it follows that the relation

$$n_\delta = \frac{C}{h\delta^\gamma}, \quad \gamma = \text{const}, \quad \gamma \in (0, 1)$$

where $C > 0$ is a constant, can be used as an *a priori* stopping rule, i.e., for such n_δ one has

$$\lim_{\delta\to 0} \|u_{n_\delta} - y\| = 0. \tag{40}$$

2.5 An iterative scheme with a stopping rule based on a discrepancy principle

In this section we assume that A is a linear finite-rank operator. Thus, it is a linear bounded operator. Let us consider equation (1) with noisy data f_δ, and a DSM of the form

$$u_{n+1} = u_n - hP(Au_n - f_\delta), \quad u_0 = u_0, \tag{41}$$

for solving this equation. Equation (41) has been used in Section 2.4. Recall that y denotes the minimal-norm solution of equation (1). Example of a choice of P is given in Section 3.

Note that $\mathcal{N} := \mathcal{N}(T) = \mathcal{N}(A)$.

Theorem 5 *Let $T := PA$, $Q := AP$. Assume that $\|Au_0 - f_\delta\| > C\delta$, $Q = Q^* \geq 0$, $T^* = T \geq 0$, $h\|T\| < 2$, $h\|Q\| < 2$, and T is a finite-rank operator. Then there exists a unique n_δ such that*

$$\|Au_{n_\delta} - f_\delta\| \leq C\delta < \|Au_{n_\delta - 1} - f_\delta\|, \quad C = const, \quad C \in (1, 2). \tag{42}$$

For this n_δ one has:

$$\lim_{\delta \to 0} \|u_{n_\delta} - y\| = 0. \tag{43}$$

Proof. Denote

$$v_n := Au_n - f_\delta, \quad w_n := u_n - y, \quad w_0 := u_0 - y.$$

From (41), one gets

$$v_{n+1} = Au_{n+1} - f_\delta = Au_n - f_\delta - hAP(Au_n - f_\delta) = v_n - hQv_n.$$

This implies

$$\begin{aligned} \|v_{n+1}\|^2 - \|v_n\|^2 &= \langle v_{n+1} - v_n, v_{n+1} + v_n \rangle \\ &= \langle -hQv_n, v_n - hQv_n + v_n \rangle \\ &= -\langle v_n, hQ(2 - hQ)v_n \rangle \leq 0 \end{aligned} \tag{44}$$

where the last inequality holds because $AP = Q \geq 0$ and $\|hQ\| < 2$. Thus, $(\|v_n\|)_{n=1}^\infty$ is a nonincreasing sequence.

Let us prove that equation (42) has a solution for $C \in (1, 2)$. One has the following commutation formulas:

$$(I - hT)^n P = P(I - hQ)^n, \quad A(I - hT)^n = (I - hQ)^n A.$$

Using these formulas, the representation

$$u_n = (I - hT)^n u_0 + h \sum_{i=0}^{n-1} (I - hT)^i P f_\delta,$$

and the identity $(I-B)\sum_{i=0}^{n-1}B^i = I-B^n$, with $B=I-hQ$, $I-B=hQ$, one gets:

$$\begin{aligned}
v_n &= Au_n - f_\delta \\
&= A(I-hT)^n u_0 + Ah\sum_{i=0}^{n-1}(I-hT)^i P f_\delta - f_\delta \\
&= (I-hQ)^n Au_0 + \sum_{i=0}^{n-1}(I-hQ)^i hQ f_\delta - f_\delta \\
&= (I-hQ)^n Au_0 - (I-(I-hQ)^n) f_\delta - f_\delta \\
&= (I-hQ)^n (Au_0 - f) + (I-hQ)^n (f-f_\delta) \\
&= (I-hQ)^n Aw_0 + (I-hQ)^n (f-f_\delta).
\end{aligned} \tag{45}$$

If $V=V^*\geq 0$ is an operator with $||V||\leq 2$, then $||I-V|| = \sup_{0\leq s\leq 2}|1-s|\leq 1$.

Note that

$$\lim_{n\to\infty}(I-hQ)^n Aw_0 = \lim_{n\to\infty} A(I-hT)^n w_0 = AP_{\mathcal{N}}w_0 = 0,$$

where $P_{\mathcal{N}}$ is the orthoprojection onto the null-space $\mathcal{N}$ of the operator T, and the continuity of A and the following relation

$$\lim_{n\to\infty}(I-hT)^n w_0 = \lim_{n\to\infty}\int_0^{\|T\|}(1-sh)^n dE_s w_0 = (E_0 - E_{-0})w_0 = P_{\mathcal{N}}w_0, \quad 0\leq sh<2,$$

were used. Therefore,

$$\lim_{n\to\infty}\|v_\delta(t)\| = \lim_{n\to\infty}\|(I-hQ)^n(f-f_\delta)\| \leq \|f-f_\delta\|\leq\delta, \tag{46}$$

where $\|I-hQ\|\leq 1$ because $Q\geq 0$ and $\|hQ\|<2$. The sequence $\{\|v_n\|\}_{n=1}^\infty$ is nonincreasing with $\|v_0\|>C\delta$ and $\lim_{n\to\infty}\|v_n\|\leq\delta$. Thus, there exists $n_\delta>0$ such that (42) holds.

Let us prove (43). Let $u_{n,0}$ be the sequence defined by the relations:

$$u_{n+1,0} = u_{n,0} - hP(Au_{n,0}-f), \quad u_{0,0}=u_0.$$

First, we have the following estimate:

$$\begin{aligned}
\|Au_{n_\delta,0}-f\| &\leq \|Au_{n_\delta} - Au_{n_\delta,0}\| + \|Au_{n_\delta}-f_\delta\| + \|f_\delta - f\| \\
&\leq \left\|\sum_{i=0}^{n_\delta-1}(I-hQ)^i hQ\right\|\|f_\delta - f\| + C\delta + \delta.
\end{aligned} \tag{47}$$

Since $0\leq hQ<2$, one has $||I-hQ||\leq 1$. This implies the following inequality:

$$\left\|\sum_{i=0}^{n_\delta-1}(I-hQ)^i hQ\right\| = \|I-(I-hQ)^{n_\delta}\|\leq 2,$$

and concludes from (47), that

$$\lim_{\delta\to 0} \|Au_{n_\delta,0} - f\| = 0. \tag{48}$$

Secondly, we claim that

$$\lim_{\delta\to 0} hn_\delta = \infty.$$

Assume the contrary. Then there exist $n_0 > 0$ and a sequence $(n_{\delta_n})_{n=1}^{\infty}$, $n_{\delta_n} < n_0$, such that

$$\lim_{n\to\infty} \|Au_{n_\delta,0} - f\| = 0. \tag{49}$$

Analogously to (44), one proves that

$$\|v_{n,0}\| \le \|v_{n-1,0}\|,$$

where $v_{n,0} = Au_{n,0} - f$. Thus, the sequence $\|v_{n,0}\|$ is nonincreasing. This and (49) imply the relation $\|v_{n_0,0}\| = \|Au_{n_0,0} - f\| = 0$. Thus,

$$0 = v_{n_0,0} = (I - hQ)^{n_0} A(u_0 - y).$$

This implies $A(u_0 - y) = (I - hQ)^{-n_0}(I - hQ)^{n_0} A(u_0 - y) = 0$, so $u_0 - y \in \mathcal{N}$. Since, by the assumption, $u_0 - y \in \mathcal{N}^\perp$, it follows that $u_0 = y$. This is a contradiction because

$$C\delta \le \|Au_0 - f_\delta\| = \|f - f_\delta\| \le \delta, \quad 1 < C < 2.$$

Thus,

$$\lim_{\delta\to 0} hn_\delta = \infty. \tag{50}$$

Let us continue the proof of (43). From (45) and $\|Au_{n_\delta} - f_\delta\| = C\delta$, one has

$$\begin{aligned} C\delta n_\delta h &= \|n_\delta h(I - hQ)^{n_\delta} Aw_0 - n_\delta h(I - hQ)^{n_\delta}(f_\delta - f)\| \\ &\le \|n_\delta h(I - hQ)^{n_\delta} Aw_0\| + \|n_\delta h(I - hQ)^{n_\delta}(f_\delta - f)\| \\ &\le \|n_\delta h(I - hQ)^{n_\delta} Aw_0\| + n_\delta h\delta. \end{aligned} \tag{51}$$

We claim that if $w_0 \in \mathcal{N}^\perp$, $0 \le hT < 2$, and T is a finite-rank operator, then

$$\lim_{\delta\to 0} n_\delta h(I - hQ)^{n_\delta} Aw_0 = \lim_{\delta\to 0} n_\delta hA(I - hT)^{n_\delta} w_0 = 0. \tag{52}$$

From (51) and (52) one gets

$$0 \le \lim_{\delta\to 0}(C - 1)\delta hn_\delta \le \lim_{\delta\to 0} \|n_\delta h(I - hQ)^{n_\delta} Aw_0\| = 0.$$

Thus,

$$\lim_{\delta\to 0} \delta n_\delta h = 0 \tag{53}$$

Now (43) follows from (50), (53) and Theorem 4. Theorem 5 is proved. □

3 Numerical experiments

3.1 Computing $u_\delta(t_\delta)$

In [3] an DSM (9) was investigated with $P = A^*$ and the SVD of A was assumed known. In general, it is computationally expensive to get the SVD of large scale matrices. In this paper, we have derived an iterative scheme for solving ill-conditioned linear algebraic systems $Au = f_\delta$ without using SVD of A.

Choose $P = (A^*A + a)^{-1}A^*$ where a is a fixed positive constant. This choice of P satisfies all the conditions in Theorem 3. In particular, $Q = AP = A(A^*A + aI)^{-1}A^* = AA^*(AA^* + aI)^{-1} \geq 0$ is a selfadjoint operator, and $T = PA = (A^*A + aI)^{-1}A^*A \geq 0$ is a selfadjoint operator. Since

$$\|T\| = \left\| \int_0^{\|A^*A\|} \frac{\lambda}{\lambda + a} dE_\lambda \right\| = \sup_{0 \leq \lambda \leq \|A^*A\|} \frac{\lambda}{\lambda + a} < 1,$$

where E_λ is the resolution of the identity of A^*A, the condition $h\|T\| < 2$ in Theorem 5 is satisfied for all $0 < h \leq 1$. Set $h = 1$ and $P = (A^*A + a)^{-1}A^*$ in (41). Then one gets the following iterative scheme:

$$u_{n+1} = u_n - (A^*A + aI)^{-1}(A^*Au_n - A^*f_\delta), \quad u_0 = 0. \tag{54}$$

For simplicity we have chosen $u_0 = 0$. However, one may choose $u_0 = v_0$ if v_0 is known to be a better approximation to y than 0 and $v_0 \in \mathcal{N}^\perp$. In iterations (54) we use a stopping rule of discrepancy type. Indeed, we will stop iterations if u_n satisfies the following condition

$$\|Au_n - f_\delta\| \leq 1.01\delta. \tag{55}$$

The choice of a affects both the accuracy and the computation time of the method. If a is too large, one needs more iterations to approach the desired accuracy, so the computation time will be large. If a is too small then the results become less accurate because for too small a the inversion of the operator $A^*A + aI$ is an ill-posed problem since the operator A^*A is not boundedly invertible. Using the idea of the choice of the initial guess of regularization parameter in [2], we choose a to satisfy the following condition:

$$\delta \leq \phi(a) := \|A(A^*A + a)^{-1}A^*f_\delta - f_\delta\| \leq 2\delta. \tag{56}$$

This can be done by using the following strategy:

1. Choose $a := \frac{\delta\|A\|^2}{3\|f_\delta\|}$ as an initial guess for a.

2. Compute $\phi(a)$. If a satisfying (56) we are done. Otherwise, we go to step 3.

3. If $c = \frac{\phi(a)}{\delta} > 3$ we replace a by $\frac{a}{2(c-1)}$ and go back to step 2. If $2 < c \leq 3$ then we replace a by $\frac{a}{2(c-1)}$ and go back to step 2. Otherwise, we go to step 4.

4. If $c = \frac{\phi(a)}{\delta} < 1$ we replace a by $3a$. If the inequality $c < 1$ has occured in some iteration before, we stop the iteration and use $3a$ as our choice for a in iterations (54). Otherwise we go back to step 2.

In our experiments, we denote by DSM the iterative scheme (54), by VR_i a Variational Regularization method (VR) with a as the regularization parameter and by VR_n the VR in which Newton's method is used for finding the regularization parameter using a discrepancy principle. We compare these methods in terms of relative error and number of iterations, denoted by n_{iter}.

All the experiments were carried in double arithmetics precision environment using MATLAB.

3.2 A linear algebraic system related to an inverse problem for the heat equation

In this section, we apply the DSM and the VR to solve a linear algebraic system used in [2]. This linear algebraic system is a part of numerical solutions to an inverse problem for the heat equation. This problem is reduced to a Volterra integral equation of the first kind with $[0,1]$ as the integration interval. The kernel is $K(s,t) = k(s-t)$ with

$$k(t) = \frac{t^{-3/2}}{2\kappa\sqrt{\pi}} \exp(-\frac{1}{4\kappa^2 t}).$$

Here, we use the value $\kappa = 1$. In this test in [2] the integral equation was discretized by means of simple collocation and the midpoint rule with n points. The unique exact solution u_n is constructed, and then the right-hand side b_n is produced as $b_n = A_n u_n$ (see [2]). In our test, we use $n = 10, 20, ..., 100$ and $b_{n,\delta} = b_n + e_n$, where e_n is a vector containing random entries, normally distributed with mean 0, variance 1, and scaled so that $\|e_n\| = \delta_{rel}\|b_n\|$. This linear system is ill-posed: the condition number of A_{100} obtained by using the function *cond* provided in MATLAB is 1.3717×10^{37}. This number shows that the corresponding linear algebraic system is severely ill-conditioned.

Table 1: Numerical results for the inverse heat equation with $\delta_{rel} = 0.05$, $n = 10i$, $i = \overline{1,10}$.

	DSM		VR_i		VR_n	
n	$\mathrm{n}_{\mathrm{iter}}$	$\frac{\|u_\delta - y\|_2}{\|y\|_2}$	$\mathrm{n}_{\mathrm{iter}}$	$\frac{\|u_\delta - y\|_2}{\|y\|_2}$	$\mathrm{n}_{\mathrm{iter}}$	$\frac{\|u_\delta - y\|_2}{\|y\|_2}$
10	3	0.1971	1	0.2627	5	0.2117
20	4	0.3359	1	0.4589	5	0.3551
30	4	0.3729	1	0.4969	5	0.3843
40	4	0.3856	1	0.5071	5	0.3864
50	5	0.3158	1	0.4789	6	0.3141
60	6	0.2892	1	0.4909	6	0.3060
70	7	0.2262	1	0.4792	8	0.2156
80	6	0.2623	1	0.4809	7	0.2600
90	5	0.2856	1	0.4816	7	0.2715
100	7	0.2358	1	0.4826	7	0.3405

Table 1 shows that the results obtained by the DSM are comparable to those by the VR_n in terms of accuracy. The time of computation of the DSM is comparable to that of the VR_n. In some situations, the results by VR_n and the DSM are the same although the

VR$_n$ uses 3 more iterations than does the DSM. The conclusion from this Table is that DSM competes favorably with the VR$_n$ in both accuracy and time of computation.

Figure 1 plots numerical solutions to the inverse heat equation for $\delta_{rel} = 0.05$ and $\delta_{rel} = 0.01$ when $n = 100$. From the figure we can see that the numerical solutions obtained by the DSM are about the same those by the VR$_n$. In these examples, the time of computation of the DSM is about the same as that of the VR$_n$.

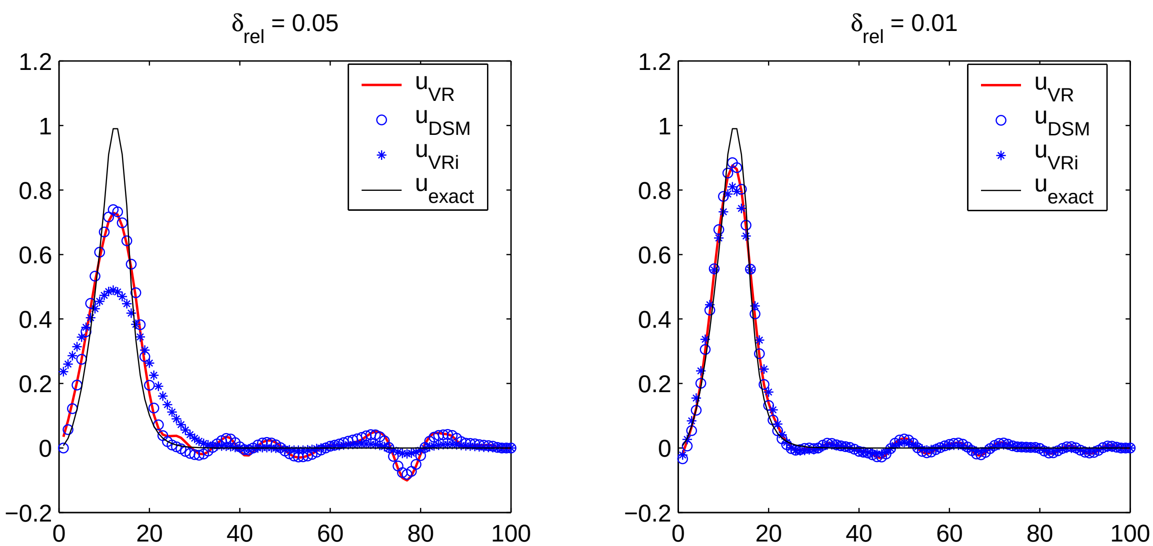

Figure 1: Plots of solutions obtained by DSM, VR for the inverse heat equation when $n = 100$, $\delta_{rel} = 0.05$ (left) and $\delta_{rel} = 0.01$ (right).

The conclusion is that the DSM competes favorably with the VR$_n$ in this experiment.

4 Concluding remark

Iterative scheme (54) can be considered as a modification the Landweber iterations. The difference between the two methods is the multiplication by $P = (A^*A + aI)^{-1}$. Our iterative method is much faster than the conventional Landweber iterations. Iterative method (54) is an analog of the Gauss-Newton method. It can be considered as a regularized Gauss-Newton method for solving ill-condition linear algebraic systems. The advantage of using (54) instead of using (4.1.3) in [2] is that one only has to compute the lower upper (LU) decomposition of $A^*A + aI$ once while the algorithm in [2] requires computing LU at every step. Note that computing the LU is the main cost for solving a linear system. Numerical experiments show that the new method competes favorably with the VR in our experiments.

References

[1] Airapetyan, R., Ramm, A. G., Dynamical systems and discrete methods for solving nonlinear ill-posed problems, Appl.Math.Reviews, vol. 1, Ed. G. Anastassiou, World Sci. Publishers, 2000, pp.491-536.

[2] Hoang, N. S. and Ramm, A. G., Solving ill-conditioned linear algebraic systems by the dynamical systems method (DSM), Inverse Problems in Sci. and Engineering, (to appear).

[3] Hoang, N. S. and Ramm, A. G., Dynamical systems gradient method for solving ill-conditioned linear algebraic systems, (submited).

[4] Ivanov, V., Tanana, V., Vasin, V., Theory of ill-posed problems, VSP, Utrecht, 2002.

[5] Lattes, J., Lions, J., Mèthode de quasi-réversibilité et applications, Dunod, Paris, 1967.

[6] Morozov, Methods of solving incorrectly posed problems, Springer Verlag, New York, 1984.

[7] Ramm, A. G., Dynamical systems method for solving operator equations, Elsevier, Amsterdam, 2007.

[8] Ramm, A. G., Dynamical systems method for solving nonlinear operator equations, International Jour. of Applied Math. Sci., 1, N1, (2004), 97-110.

[9] Ramm, A. G., Dynamical systems method for solving operator equations, Communic. in Nonlinear Sci. and Numer. Simulation, 9, N2, (2004), 383-402.

[10] Ramm, A. G., Discrepancy principle for the dynamical systems method, Communic. in Nonlinear Sci. and Numer. Simulation, 10, N1, (2005), 95-101

[11] Ramm, A. G., Dynamical systems method (DSM) and nonlinear problems, in the book: Spectral Theory and Nonlinear Analysis, World Scientific Publishers, Singapore, 2005, 201-228. (ed J. Lopez-Gomez).

[12] Ramm, A. G., Dynamical systems method (DSM) for unbounded operators, Proc.Amer. Math. Soc., 134, N4, (2006), 1059-1063.

[13] Tautenhahn, U., On the asymptotical regularization of nonlinear ill-posed problems, Inverse Problems, 10 (1994) 1405-1418.

[14] Vainikko, G., Veretennikov, A., Iterative processes in ill-posed problems, Nauka, Moscow, 1996.

[15] Vasin, V., Ageev, A., Ill-posed problems with a priori information, Nauka, Ekaterinburg, 1993.